The Variational-Iteration Method to Solve the Nonlinear Boltzmann Equation

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Z. Naturforsch. **63a,** 131 – 139 (2008); received September 18, 2007

The time-dependent nonlinear Boltzmann equation, which describes the time evolution of a single-particle distribution in a dilute gas of particles interacting only through binary collisions, is considered for spatially homogeneous and inhomogeneous media without external force and energy source. The nonlinear Boltzmann equation is converted to a nonlinear partial differential equation for the generating function of the moments of the distribution function. The variational-iteration method derived by He is used to solve the nonlinear differential equation of the generating function. The moments for both homogeneous and inhomogeneous media are calculated and represented graphically as functions of space and time. The distribution function is calculated from its moments using the cosine Fourier transformation. The distribution functions for the homogeneous and inhomogeneous media are represented graphically as functions of position and time.

Key words: Time-Dependent Nonlinear Boltzmann Equation; Homogeneous and Inhomogeneous Media; Moments of Distribution Function; Variational-Iteration Method.

1. Introduction

More than a century ago, Boltzmann derived the original transport equation to describe the time evolution of a one-particle distribution function in a dilute gas of particles interacting only through binary collisions [1, 2]. The transport theory has become an important topic in physics and engineering, since particle transport processes arise in a wide variety of physical phenomena.

Because of the complex structure of the collision term, this integro-differential equation resists a strong solution in general. The exact solutions of the nonlinear Boltzmann equation have been found only for special model cases. The most important stimulus undoubtedly came from the discovery of an exact solution of the nonlinear Boltzmann equation by using the similarity method for Maxwell molecules, found independently by Bobylev [3] and by Krook and Wu [4] (BKW model). The possible conjecture of Krook and Wu was that a significant class of initial distributions may relax rapidly to the BKW model, which then evolves essentially unchanged to the final equilibrium. Many authors have presented numerical [5] and analytical [6 – 10] evidence against the validity of this conjecture. Another important development was the Laguerre series solution of the nonlinear Boltzmann equation for Maxwell molecules and the Maxwell-type model in general [11]. Convergence properties for this model were first established by Barnsly and Cornille [6, 12]. Nonnenmacher found the exact similarity solutions of the nonlinear Boltzmann equation in a homogeneous medium [13, 14]. For arbitrary initial conditions, Ernst and Hendriks [15, 16] obtained a solution applying the Laplace transformation. Also Schürrer and Schaler [17] obtained the exact solution of the Boltzmann equation (linear and nonlinear) in the very hard particles (VHP) model with removal interaction for arbitrary initial distributions. Koller et al. and Schürrer [18, 19] found an approximate solution for the scalar nonlinear Boltzmann equation in its multi-group representation. El-Wakil et al. [20] found an exact analytical solution for the spatially inhomogeneous Boltzmann equation by resorting to the Nikolskii method [21].

There are many different methods to solve the nonlinear Boltzmann equation. In this paper, we will solve this equation using the variational-iteration method (VIM) [22 – 31].

The VIM is a semi-analytical method using a general Lagrange multiplier, which can be determined optimally by the variational theory [32]. This method was first proposed by He [22 – 25]. It has been used to solve effectively, easily and accurately a large class of

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nonlinear problems. The approximate solutions by the VIM converge rapidly to accurate solutions [26–31].

In this paper, the problem of particle transport in a host medium without external forces and energy sources in view of the two cases of homogeneous and inhomogeneous media is considered and solved using the VIM. The calculations of the solutions are carried out for space and time and represented graphically. The skeleton of the paper is as follows: Section 2 contains the description of the VIM. The formulation of the problem is given in Section 3. In Section 4, the solution of the nonlinear differential equations of the problem using the VIM is given. We present our numerical calculations and results in Section 5. The conclusion of this paper is given in Section 6.

2. The Variational-Iteration Method

The VIM is a modified general Lagrange multiplier method [32, 33]. The main feature of the method is that the solution of a mathematical problem with linear assumption is used as initial approximation or trial function. Then a more precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution and is described as follows.

Consider the general nonlinear equation

$$\hat{L}U(x)+\hat{N}U(x)=g(x), \tag{1}$$

where $\hat{L}$ is the linear operator part while $\hat{N}$ is the nonlinear operator part and assume that $U_0(x)$ is the solution of the linear homogeneous equation

$$\hat{L}u_0(x)=0. \tag{2}$$

He [22–25] has modified this method into an iteration method to correct the value of some special point (x) as follows:

$$\begin{aligned}U_{n+1}(x)=U_n(x)+\int_0^x \mathrm{d}y\lambda(y)[\hat{L}U_n(y)\\+\hat{N}\tilde{U}_n(y)-g(y)],\quad n\geq 0,\end{aligned} \tag{3}$$

where $\lambda(y)$ is a general Lagrange's multiplier, which can be identified optimally via the variational theory [22, 23], the subscript n denotes the n-th order approximation and $\tilde{U}_n(x)$ is considered as a restricted variation function, i. e. $\delta\tilde{U}_n(x)=0$.

The above technique will be used to solve the nonlinear Boltzmann equation for single-particle distribution in a dilute gas in a homogeneous and an inhomogeneous medium with different boundary conditions.

3. Formulation of the Problem

The time evolution of a one-particle distribution in a gas of particles interacting through binary collisions is described by the nonlinear Boltzmann equation [2, 4]

$$\left[\frac{\partial}{\partial t}+\underline{v}\cdot\underline{\nabla}_r+\frac{1}{m}\underline{F}\cdot\underline{\nabla}_v\right]f(\underline{r},\underline{v},t)=C(f,f)+Q(\underline{v},t), \tag{4}$$

where $f(\underline{r},\underline{v},t)$ is the single-particle velocity distribution function, $\underline{F}$ some external force, $Q(\underline{v},t)$ a source term and $C(f,f)$ the nonlinear collision term that can be represented in the form [2, 4]

$$\begin{aligned}C(f,f)=\int\mathrm{d}\underline{w}\int\mathrm{d}\Omega gI(g,\theta)\Big[f(\underline{r},\underline{v}',t)f(\underline{r},\underline{w}',t)\\-f(\underline{r},\underline{v},t)f(\underline{r},\underline{w},t)\Big],\end{aligned} \tag{5}$$

where $\underline{v}$ and $\underline{w}$ are the initial velocities, $\underline{v}'$ and $\underline{w}'$ are the final velocities and $g=|\underline{v}-\underline{w}|$ is the relative velocity. In the binary collision process, the initial and final velocities are related by the following dynamics:

$$\begin{aligned}\underline{v}'&=\frac{1}{2}\left[(\underline{v}+\underline{w})+|\underline{v}-\underline{w}|\hat{\underline{\Omega}}\right]\\ \text{and}\quad \underline{w}'&=\frac{1}{2}\left[(\underline{v}+\underline{w})-|\underline{v}-\underline{w}|\hat{\underline{\Omega}},\right],\end{aligned} \tag{6}$$

where $\hat{\underline{\Omega}}=\frac{(\underline{v}'-\underline{w}')}{|\underline{v}'-\underline{w}'|}$ is the direction of the scattering in the relative coordinate frame. Here, $\mathrm{d}\Omega=\sin(\theta)\,\mathrm{d}\theta\,\phi$, $\theta\in[0,\pi]$ is the scattering angle, $\phi\in[0,2\pi]$ is the azimuthal angle, which determines the orientation of the plane of scattering and $I(g,\theta)$ is the differential scattering cross-section.

For Maxwell molecules and isotropic scattering, the differential scattering cross-section has, in the centre of mass system, the form [13, 14]

$$gI(g,\theta)=c_0, \tag{7a}$$

where c_0 is a constant. Therefore, the collision term $C(f,f)$, in d-dimensions, can be rewritten as

$$\begin{aligned}C(f,f)=-2(2\pi)^{(d-1)/2}c_0\Big[M_0(r,t)f(r,\underline{v},t)\\-\int\mathrm{d}\underline{w}f(\underline{r},\underline{v}',t)f(\underline{r},\underline{w}',t)\Big],\end{aligned} \tag{7b}$$

where $M_0(r,t)$ is the zero-order moment of the distribution function, which is the particle number density of the system $[n(r,t)]$. The n-th order normalized moment $M_n(r,t)$ in d-dimensions can be defined by

$$M_n(r,t) = \frac{\Gamma(d/2)}{2^n\Gamma(n+d/2)} \int \mathrm{d}\underline{v} v^{2n} f(r,\underline{v},t), \quad (8)$$

where $\Gamma(X)$ is the gamma function of argument (x) and $d = 1$ or 3.

The distribution function $f(r,\underline{v},t)$ can be calculated using the moments $M_n(r,t)$ by assuming

$$\Phi(r,\underline{v},t) = v^{d-1} f(r,\underline{v},t), \quad (9a)$$

that has the cosine Fourier transformation as

$$\tilde{\Phi}(r,p,t) = 2\int_0^\infty \mathrm{d}\underline{v} \cos(pv)\Phi(r,\underline{v},t). \quad (9b)$$

The inverse Fourier transformation is given by

$$\Phi(r,\underline{v},t) = \frac{1}{\pi}\int_0^\infty \mathrm{d}p \cos(pv)\tilde{\Phi}(r,p,t). \quad (9c)$$

Using the series expansion of the cosine function in the Fourier transformation (9b) with the definitions (9a) and (8) and some manipulations leads to

$$\tilde{\Phi}(r,p,t) = \frac{1}{(2\pi)^{(d-1)/2}} \sum_{n=0}^\infty (-1)^n \frac{(p^2/2)^n}{n!} \cdot [(d-1)n+1]M_n(r,t), \quad (10)$$

where the vector velocity element of integration in the isotropic scattering d-dimensions medium is given by $\mathrm{d}\underline{v} = 2(2\pi v^2)^{(d-1)/2}\,\mathrm{d}v$.

The inverse Fourier transformation of (10) represents the distribution function $f(r,\underline{v},t)$ in the form

$$f(r,v,t) = \frac{1}{(2\pi v^2)^{(d-1)/2}} \frac{1}{\pi}\int_0^\infty \mathrm{d}p \cos(pv) \cdot \sum_{n=0}^\infty \frac{(-1)^n}{n!}\left(\frac{p^2}{2}\right)^n [(d-1)n+1]M_n(r,t). \quad (11)$$

In this paper, we will introduce the cases of single-particle distribution in a dilute gas in spatially homogeneous and inhomogeneous media without external forces and energy sources, i. e. $\underline{F} = 0$ and $Q = 0$.

3.1. Spatially Homogeneous Medium

The nonlinear Boltzmann equation with no external force and no energy source in a spatially homogeneous medium, i. e. $\underline{\nabla}_r f(\underline{r},\underline{v},t) = 0$, becomes

$$\frac{\partial}{\partial t} f(\underline{v},t) = C(f,f). \quad (12a)$$

Substituting (7b) into (12a) for space-independent functions yields

$$\frac{\partial}{\partial t} f(\underline{v},t) + 4\pi c_0 M_0(t) f(\underline{v},t) = 4\pi c_0 \int \mathrm{d}\underline{w} f(v',t) f(\underline{w}',t). \quad (12b)$$

Multiplying this equation by $(\underline{v}\cdot\underline{v})^n = v^{2n}$ and normalizing the integral leads to a nonlinear equation for the energy moments [4, 13] of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} M_n(t) + 4\pi c_0 M_0(t) M_n(t) = \frac{4\pi c_0}{n+1} \sum_{m=0}^n M_m(t) M_{n-m}(t). \quad (13a)$$

Putting the dimensionless time $\tau = 4\pi c_0 t \to t$, one gets

$$\frac{\mathrm{d}}{\mathrm{d}t} M_n(t) + M_0(t) M_n(t) = \frac{1}{n+1} \sum_{m=0}^n M_m(t) M_{n-m}(t). \quad (13b)$$

Introducing now the Krook and Wu [4] generating function for the moments as

$$G(\omega,t) = \sum_{n=0}^\infty \omega^n M_n(t), \quad (14)$$

multiplying (13b) by ω^n and subsequent summation over all n leads to [13]

$$\frac{\partial}{\partial t} G(\omega,t) + M_0(t) G(\omega,t) = \frac{1}{\omega}\int_0^\omega \mathrm{d}\omega' G^2(\omega',t). \quad (15a)$$

Multiplying this equation through ω and differentiating with respect to ω leads to

$$\frac{\partial^2}{\partial t\partial\omega}[\omega G(\omega,t)] + M_0(t)\frac{\partial}{\partial\omega}[\omega G(\omega,t)] = G^2(\omega,t). \quad (15b)$$

Introducing the transformations

$$\eta = \frac{(1-\omega)}{\omega} \tag{16a}$$

and

$$u(\eta,t) = \omega G(\omega,t), \tag{16b}$$

where

$$\frac{\partial}{\partial\eta} = -\omega^2 \frac{\partial}{\partial\omega}, \tag{16c}$$

and substituting these transformations into (15b) yields

$$\frac{\partial^2}{\partial t\partial\eta}u(\eta,t) + M_0(t)\frac{\partial}{\partial\eta}u(\eta,t) + u^2(\eta,t) = 0. \tag{17a}$$

From (13b) one obtains $M_0(t) = \text{constant} = 1$. Therefore, the above equation can be rewritten as

$$\frac{\partial}{\partial\eta}u(\eta,t) + \frac{\partial^2}{\partial t\partial\eta}u(\eta,t) + u^2(\eta,t) = 0. \tag{17b}$$

This is a nonlinear differential equation, which can be solved to give the generating function of the moments of the distribution function. This generating function can be expanded to give the moments. The moments can be used to calculate the distribution function of the nonlinear Boltzmann equation (12).

3.2. Spatially Inhomogeneous Medium

The one-dimensional nonlinear Boltzmann equation with no external force and no internal energy source for a spatially inhomogeneous medium has the form

$$\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right] f(x,v,t) = C(f,f), \tag{18a}$$

where the collision term $C(f,f)$ is defined by (7b) replacing r by x.

Substituting (7b) into (18a) yields the nonlinear Boltzmann equation

$$\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + 2c_0 M_0(x,t)\right] f(x,v,t) = 2c_0 \int \mathrm{d}w f(x,v',t). \tag{18b}$$

Multiplying this equation by v^{2n} times the normalization factor and integrating over $v \in [0,\infty)$ leads to a nonlinear equation for the moments $M_n(x,t)$ as

$$\frac{\partial}{\partial t}M_n(x,t) + \frac{\partial}{\partial x}M_{n+1/2}(x,t) + 2c_0 M_0(x,t)M_n(x,t) = \frac{2c_0}{(n+1)}\sum_{m=0}^{n} M_m(x,t)M_{n-m}(x,t), \tag{19a}$$

where the moments $M_n(x,t)$ are defined by (8).

Putting the dimensionless quantities $\tau = 2c_0 t \to t$ and $\xi = 2c_0 x \to x$, one gets the dimensionless moment equation

$$\frac{\partial}{\partial t}M_n(x,t) + \frac{\partial}{\partial x}M_{n+1/2}(x,t) + M_0(x,t)M_n(x,t) = \frac{1}{(n+1)}\sum_{m=0}^{n} M_m(x,t)M_{n-m}(x,t). \tag{19b}$$

Multiplying this equation by ω^n and subsequent summation over all n gives

$$\sum_{n=0}^{\infty}\omega^n\frac{\partial}{\partial t}M_n(x,t) + \sum_{n=0}^{\infty}\omega^n\frac{\partial}{\partial t}M_{n+1/2}(x,t) + \sum_{n=0}^{\infty}\omega^n M_0(x,t)M_n(x,t) = \sum_{n=0}^{\infty}\frac{\omega^n}{(n+1)}\sum_{m=0}^{n} M_m(x,t)M_{n-m}(x,t). \tag{20}$$

Introducing the Krook-Wu [4] generating function for the moments defined by (14) leads to the nonlinear integro-differential equation

$$\frac{\partial}{\partial t}G(\omega,x,t) + \frac{\partial}{\partial x}G(\omega,x,t) + M_0(x,t)G(\omega,x,t) = \frac{1}{\omega}\int_0^{\omega} \mathrm{d}\omega' G^2(\omega',x,t). \tag{21a}$$

Multiplying this equation by ω and differentiation with respect to ω leads to

$$\frac{\partial^2}{\partial\omega\partial t}[\omega G] + \frac{\partial^2}{\partial\omega\partial x}[\omega G] + M_0(x,t)\frac{\partial}{\partial\omega}[\omega G] = G^2(\omega,x,t). \tag{21b}$$

Substituting the transformations given by (16) into the last equation yields

$$\frac{\partial^2}{\partial\eta\partial t}u(\eta,x,t) + \frac{\partial^2}{\partial\eta\partial x}u(\eta,x,t) + M_0(x,t)\frac{\partial}{\partial\eta}u(\eta,x,t) + u^2(\eta,x,t) = 0. \tag{22a}$$

For this system, the zero-order moment $M_0(x,t) =$ constant $= 1$, which is just the total number of particles conservation law, this equation can be rewritten as

$$\frac{\partial}{\partial\eta}u(\eta,x,t)+\frac{\partial^2}{\partial\eta\partial t}u(\eta,x,t) + \frac{\partial^2}{\partial\eta\partial x}u(\eta,x,t)+u^2(\eta,x,t)=0. \tag{22b}$$

This equation is a nonlinear differential equation that describes the generating function of the moments of the nonlinear Boltzmann equation (18).

Equations (17) and (22) will be solved using the VIM [22–31] to give the generating function of the distribution function moments that can be used to calculate the moments and then the distribution function of a single particle in a dilute gas.

4. Solution of the Problem

The nonlinear differential equations (17) and (22), which describe the generating functions of the moments of the nonlinear Boltzmann equation solution, will be solved using the VIM [22–31]. The two cases of distribution in homogeneous and inhomogeneous media will be considered.

4.1. The Homogeneous Medium

The correction functional for (17) can be written as [22–25]

$$u_{n+1}(\eta,t)=u_n(\eta,t)+\int_0^{\eta}\mathrm{d}\eta'\lambda(\eta') \cdot\left[\frac{\partial}{\partial\eta'}u_n+\frac{\partial^2}{\partial\eta'\partial t}\tilde{u}_n+\tilde{u}_n^2(\eta',t)\right], \quad n\geq 0, \tag{23}$$

with the zero-order approximation $u_0(\eta,t)$ given by the condition $u(0,t)$ as

$$u_0(\eta,t)=u(0,t), \tag{24}$$

where $\lambda(\eta')$ is the Lagrange multiplier that is identified by taking the variation of (23), and $\tilde{u}_n(\eta,t)$ is considered as a restricted variation, i. e. $\delta\tilde{u}_n=0$. Therefore, the variation of (23) gives

$$\delta u_{n+1}(\eta,t)=\delta u_n(\eta,t) + \delta\int_0^{\eta}\mathrm{d}\eta'\lambda(\eta')\frac{\partial}{\partial\eta'}u_n(\eta',t). \tag{25a}$$

As the correction functional is stationary and $\delta u_n(0,t)=0$,

$$\delta u_n(\eta,t)+\lambda(\eta')\delta u_n(\eta',t)|_{\eta'=\eta} - \int_0^{\eta}\mathrm{d}\eta'\frac{\mathrm{d}\lambda}{\mathrm{d}\eta'}\delta u_n(\eta',t)=0. \tag{25b}$$

This equation yields the following stationary conditions:

$$1+\lambda|_{\eta'=\eta}=0 \tag{26a}$$

and

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\eta}=0. \tag{26b}$$

Solving this system, the Lagrange multiplier can be identified by

$$\lambda(\eta)=-1. \tag{26c}$$

The iteration formula (23) becomes

$$u_{n+1}(\eta,t)=u_n(\eta,t) -\int_0^{\eta}\mathrm{d}\eta'\left[\frac{\partial}{\partial\eta'}u_n+\frac{\partial^2}{\partial\eta'\partial t}u_n+u_n^2(\eta',t)\right], \quad n\geq 0. \tag{27}$$

Beginning with the constant boundary condition

$$u_0(\eta,t)=u(0,t)=c \tag{28a}$$

and using this zero-order approximation in the correctional-iteration formula (27) leads to the first-order approximation in the form

$$u_1(\eta,t)=c-c^2\eta. \tag{28b}$$

This leads to the second-order approximation

$$u_2(\eta,t)=c-c^2\eta+c^3\eta^2-c^4\eta^3/3. \tag{28c}$$

Also, the third-order approximation is given as

$$u_3(\eta,t)=c-c^2\eta+c^3\eta^2-c^4\eta^3+(2/3)c^5\eta^4 -c^6\eta^5/3+c^7\eta^6/9-c^8\eta^7/63. \tag{28d}$$

By the same way one can obtain the n-th order approximation as

$$u_n(\eta,t)=c-c^2\eta+c^3\eta^2-c^4\eta^3+c^5\eta^4\mp\dots \tag{28e}$$

As n tends to infinity, this formula leads to

$$u(\eta,t) = \lim_{n\to\infty} u_n(\eta,t) = \frac{c}{(1+c\eta)}, \tag{29}$$

which is the solution of (17) in a closed form. Substituting (16) into this solution gives the generating function

$$G(\omega,t) = \frac{c}{[\omega(1-c)+c]}. \tag{30}$$

Substituting into (14) gives the different moments

$$M_n(t) = \left(1 - \frac{1}{c}\right)^n. \tag{31}$$

From this equation, one gets the zero-order moment $M_0(t) = 1$, which yields the total particles number and $M_1(t) = \text{constant} = (1 - \frac{1}{c})$, which represents the total energy flux of the system. These two moments represent the conservation of both the total particle number and the total energy for a single particle in a dilute gas of Maxwell particles and isotropic scattering homogeneous medium [2, 4, 13, 14].

The distribution function, $f(\underline{v},t)$ of a single particle in a dilute gas in a 3-dimensional homogeneous medium is given by substituting (31) into (11) as follows:

$$f(\underline{v},t) = \frac{1}{2\pi^2 v^2} \int_0^\infty \mathrm{d}p \cos(pv) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{p^2}{2}\right)^n (2n+1) \left(1 - \frac{1}{c}\right)^n. \tag{32a}$$

After some manipulations this formula leads to the distribution function

$$f(\underline{v},t) = \left(\frac{1}{2\pi K}\right)^{3/2} \exp\left(-\frac{v^2}{2K}\right), \tag{32b}$$

where $K = (c-1)/c$ and c is an arbitrary constant.

4.2. The Inhomogeneous Medium

The correctional-iteration equation for (22) can be written as [22–25]

$$u_{n+1}(\eta,x,t) = u_n(\eta,x,t) + \int_0^\eta \mathrm{d}\eta' \lambda(\eta') \cdot \left[\frac{\partial}{\partial\eta'} u_n + \frac{\partial^2}{\partial\eta'\partial t} \tilde{u}_n + \frac{\partial^2}{\partial\eta'\partial x} \tilde{u}_n + \tilde{u}_n^2(\eta',x,t)\right], \quad n \geq 0, \tag{33}$$

with zero-order approximation

$$u_0(\eta,x,t) = u(0,x,t), \tag{34}$$

where $\lambda(\eta')$ is a Lagrange's multiplier that is identified by taking the variational of (33). Here $\tilde{u}_n(\eta,x,t)$ is considered as a restricted variation, i. e. $\delta\tilde{u}_n = 0$. The variation of (33) is given by

$$\delta u_{n+1}(\eta,x,t) = \delta u_n(\eta,x,t) + \delta \int_0^\eta \mathrm{d}\eta' \lambda(\eta') \frac{\partial}{\partial\eta'} u_n(\eta',x,t). \tag{35a}$$

Therefore, as the correction functional is stationary and $\delta u_n(0,x,t) = 0$, one has

$$\delta u_n(\eta,x,t) + \lambda(\eta') \delta u_n(\eta',x,t)|_{\eta'=\eta} - \int_0^\eta \mathrm{d}\eta' \frac{\mathrm{d}\lambda}{\mathrm{d}\eta'} \delta u_n(\eta',x,t) = 0, \tag{35b}$$

which leads to the stationary conditions

$$1 + \lambda(\eta')|_{\eta'=\eta} = 0 \tag{36a}$$

and

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\eta} = 0. \tag{36b}$$

Solving this system, the Lagrange multiplier can be identified by

$$\lambda(\eta) = -1. \tag{36c}$$

Therefore, the correctional-iteration formula (33) becomes

$$u_{n+1}(\eta,x,t) = u_n(\eta,x,t) - \int_0^\eta \mathrm{d}\eta' \cdot \left[\frac{\partial}{\partial\eta'} u_n + \frac{\partial^2}{\partial\eta'\partial t} u_n + \frac{\partial^2}{\partial\eta'\partial x} u_n + u_n^2(\eta',x,t)\right], \quad n \geq 0. \tag{37}$$

Beginning with the condition

$$u(0,x,t) = \exp(x-t), \tag{38}$$

the zero-order approximation is given by

$$u_0(\eta,x,t) = \exp(x-t). \tag{39a}$$

Using this zero-order approximation in the correctional-iteration formula (37), the first-order approximation is given in the form

$$u_1(\eta,x,t)=\exp(x-t)-\eta\exp[2(x-t)]. \quad (39b)$$

Using the first approximation of (37), the second approximation is given as

$$u_2(\eta,x,t)=\mathrm{e}^{(x-t)}-\eta\mathrm{e}^{[2(x-t)]}+\eta^2\mathrm{e}^{[3(x-t)]} -\eta^3\mathrm{e}^{[4(x-t)]}/3. \quad (39c)$$

The third approximation is given as

$$\begin{aligned}u_3(\eta,x,t)=&\,\mathrm{e}^{(x-t)}-\eta\mathrm{e}^{2(x-t)}+\eta^2\mathrm{e}^{3(x-t)}\\&-\eta^3\mathrm{e}^{4(x-t)}+(2/3)\eta^4\mathrm{e}^{5(x-t)}\\&-\eta^5\mathrm{e}^{6(x-t)}/3+\eta^6\mathrm{e}^{7(x-t)}/9\\&-\eta^7\mathrm{e}^{8(x-t)}/63.\end{aligned} \quad (39d)$$

In the same way, one can obtain the n-th order approximation as

$$u_n(\eta,x,t)=\exp(x-t)-\eta\exp[2(x-t)] +\eta^2\exp[3(x-t)]\mp\ldots. \quad (39e)$$

As n tends to infinity, this leads to the solution of (22) using the VIM in the closed form:

$$u(\eta,x,t)=\lim_{n\to\infty}u_n(\eta,x,t)=\frac{\exp(x-t)}{[1+\eta\exp(x-t)]}. \quad (40)$$

Substituting (16) into this equation leads to the generating function of the moments

$$G(\omega,x,t)=\frac{\exp(x-t)}{[\omega+(1-\omega)\exp(x-t)]}. \quad (41)$$

Substituting into (14), the different moments $M_n(x,t)$ of the one-particle distribution function in an inhomogeneous medium is given in the form

$$M_n(x,t)=\{1-\exp[-(x-t)]\}^n. \quad (42)$$

The total particles number $M_0(x,t)$ and the total energy flux $M_1(x,t)$ of the system have the forms

$$M_0(x,t)=1 \quad (43a)$$

and

$$M_1(x,t)=\{1-\exp[-(x-t)]\}. \quad (43b)$$

These relations show that the total particles number is conservative while the total energy is not conservative for single-particle distribution in an inhomogeneous medium of a dilute Maxwell gas.

The distribution function $f(x,v,t)$ of a single particle in an inhomogeneous medium of a dilute Maxwell gas is given by substituting (42) into (11) and using some mathematical manipulations as

$$f(x,v,t)=\left[\frac{1}{2\pi K(x,t)}\right]^{1/2}\exp\left[-\frac{v^2}{2K(x,t)}\right], \quad (44)$$

where $K(x,t)=1-\exp[-(x-t)]$.

5. Results and Discussion

The nonlinear Boltzmann equation has become an important topic in physics and engineering, since particle transport processes arise in a wide variety of physical phenomena. In this paper, the nonlinear Boltzmann equation converts to a nonlinear partial differential equation that is solved using He's variational-iteration method. The nonlinear Boltzmann equation is solved for no external force, $\underline{F}=0$, and no energy source, $Q=0$, in the two cases of homogeneous and inhomogeneous media.

In the case of a spatially homogeneous medium, using constant boundary conditions, all the energy moments are constants for space and time. This verifies the conservation of both the number of particles density $n(t)=M_0$ and the energy density $E(t)=M_1$ of the system. The distribution function $f(v,t)$ is given as a function of the velocity only and does not depend on

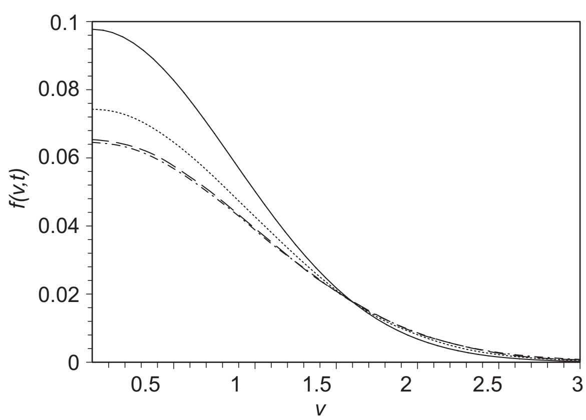

Fig. 1. The distribution function of a single particle in a dilute gas in a homogeneous medium for the values: — $c=4$; ··· $c=10$; ---- $c=50$; -·-·-·- $c=100$.

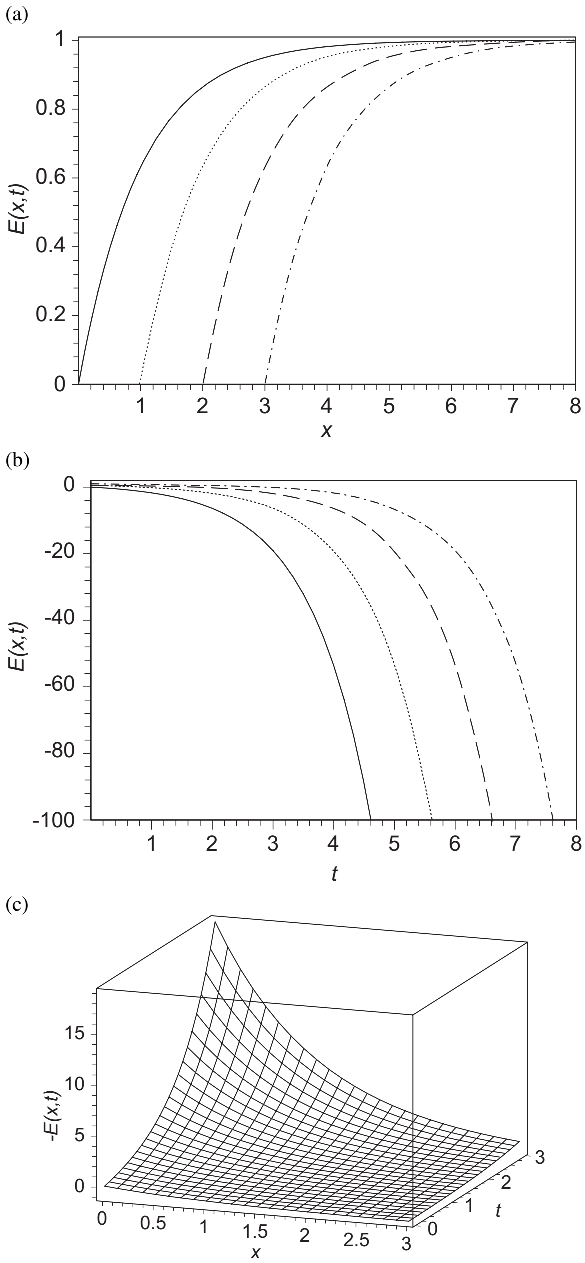

Fig. 2. The total energy density $E(x,t)$ of a single particle in a dilute gas as a function of the space (x) and time (t) in an inhomogeneous medium: (a) $E(x,t)$ against x at different values of t: — $t=0$; ··· $t=1$; ---- $t=2$; -·-·-·- $t=3$. (b) $E(x,t)$ against t at different values of x: — $x=0$; ··· $x=1$; ---- $x=2$; -·-·-·- $x=3$. (c) 3-dimensional plot of $E(x,t)$ against position (x) and time (t).

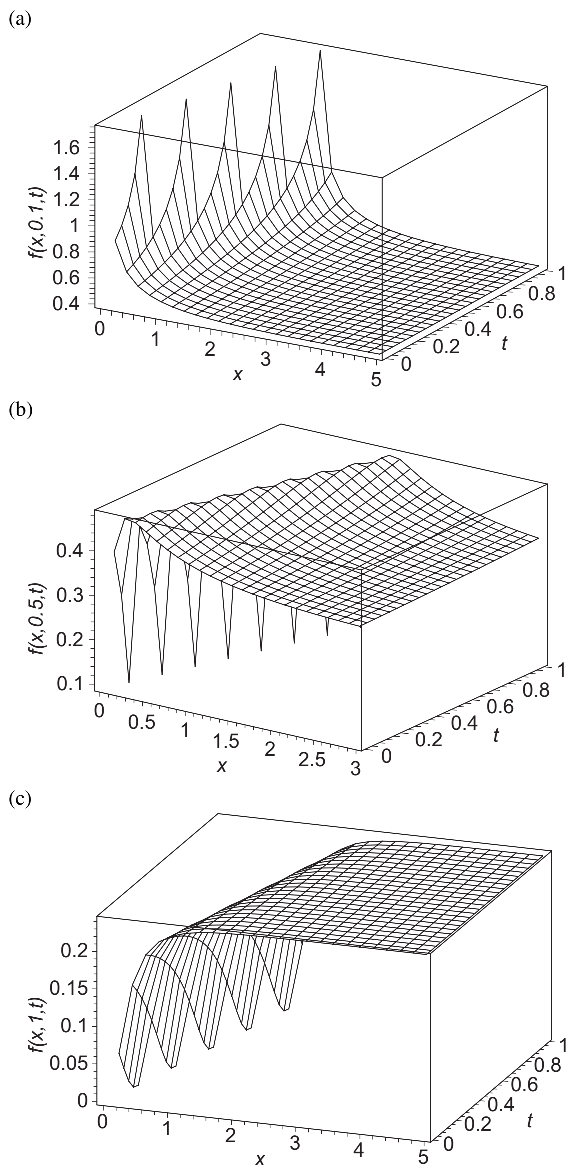

Fig. 3. The distribution function $f(x,v,t)$ of a single particle in a dilute gas in an inhomogeneous medium at the velocities: (a) $v=0.1$; (b) $v=0.5$; (c) $v=1$.

the space and time. The distribution function $f(v,t)$ is drawn as a function of the velocity (v) for different values of the constant (c) in Figure 1.

In a spatially inhomogeneous medium, using the exponential function of space and time, $\exp(x-t)$, the zero moment equals to unity and the values of the first and other moments depend on time and position. This verifies the conservation law of the number of particles density of this system while the energy density of this system is not conserved. Figure 2a shows the relation between the energy density $E(x,t)$ and the position and time as a 3-dimensional graph. Figure 2b shows the energy density as a function of time (t) at different position values ($x = 0$, 1, 2 and 3). The results show that the energy density of the system decays with time. The relation between $E(x,t)$ and position (x) at different values of time ($t = 0$, 1, 2 and 3) is presented in Figure 2c. The energy density of the system increases with the position until its value reaches unity. The distribution function $f(x,v,t)$ is represented as a function of the position (x) and time (t) at different velocities. The distribution function at the velocities $v = 0.1$, 0.5 and 1 are given in Figs. 3a, b and c, respectively.

6. Conclusion

The nonlinear integro-differential Boltzmann equation can be transformed into a nonlinear differential equation for the generating function of the energy moments of the distribution function. The resulting nonlinear differential equation is solved using the asymptotic method, which is called VIM [22–31]. The solutions are used to describe the distribution of a single particle with a dilute Maxwell isotropic gas in a homogeneous and an inhomogeneous medium. For the homogeneous medium with constant boundary condition, all the velocity moments are constants and the distribution function depends only on the velocity independent upon position and time. For the inhomogeneous medium with exponential boundary condition, the particles density is conserved while the energy density is not conserved but depends on both the position and time. The distribution function in this case is a function dependent upon all position, velocity and time.

Acknowledgement

The authors would like to express their great thankfulness to Prof. S. A. El-Wakil for his suggestion and review of the research and for his encouragement and supervision.

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